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Spherical symmetry of generalized EYMH fields

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Abstract

The possible actions of symmetry groups on generalized Higgs fields coupled to an Einstein–Yang–Mills field are studied with differential geometrical techniques involving principal and associated bundles. A classification of conjugacy classes of these actions and the form of the corresponding invariant Einstein–Yang–Mills–Higgs (EYMH) fields is obtained and then applied to the case of static spherically symmetric fields over four-dimensional space-time. We identify the representations of the gauge group for which spherically symmetric Higgs fields exist. Then the set of all field equations for the independent functions that describe these fields is analyzed and the corresponding ordinary system of differential equations is derived and shown to be consistent.

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1. Introduction

It has long been realized [1] that Yang–Mills potentials correspond to the local functions needed to describe a connection on a principal bundle P over space-time M whose structure group G_o is the physical gauge group. A local gauge transformation is then represented by a change to another local section of P.

Similarly, while the standard Higgs field is a scalar function on space-time with values in the Lie algebra $\mathfrak{g}_o$ of G_o and transforms under the adjoint transformation in $\mathfrak{g}_o$, it is easily generalized to have values in a vector space (or manifold) on which the gauge group acts. In fact, a *generalized*

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Higgs field is best defined as a section of a bundle E associated to P since that already incorporates the relation to the gauge group and the gauge changes [2–5].

Over the last one or two decades much work has been done exploring special classical solutions of Yang–Mills gauge fields in interaction with the gravitational field, the so-called Einstein–Yang–Mills (EYM) fields. This work starting with numerical regular and black hole solutions for the $SU(2)$ gauge group [6–9] and followed soon by rigorous existence proofs [10–12] has dealt mainly with the spherically symmetric static case although some very special rotationally symmetric stationary solutions have also been numerically constructed. In most papers the gauge group was chosen to be the simplest nonabelian one, namely $SU(2)$, but some numerical studies have been done for $SU(n)$ with $n \geq 3$. The inclusion of gravity unfortunately makes some of the techniques used in the Yang–Mills–Higgs theories on Minkowski space less effective like, for example, the Bogomol'nyi equations which have lead to many rigorous results for arbitrary compact gauge groups (cf. [13]), for a survey of EYM solutions see [14].

We have been particularly interested in studying the general geometric, analytic and algebraic problems that arise when the gauge group is an arbitrary compact semisimple Lie group and the symmetry group acts on the principal bundle by arbitrary automorphisms as long as they project onto the 'normal' action of $SO(3)$ by isometries on a static space-time manifold. In [15,16] it was shown how the conjugacy classes of these group actions correspond to Dynkin's [17] classification of $\mathfrak{sl}_2\mathbb{C}$ subalgebras of semisimple Lie algebras. We then derived and analyzed to some extent the resulting system of ordinary differential equations [16,18,19]. It turns out that the analysis of these equations even in this static spherically symmetric case poses already many interesting problems. About the set of global solutions satisfying appropriate physical boundary conditions still very little is known for gauge groups other than $SU(2)$. The possible actions of $SU(2)$ by automorphisms on principal bundles come in two main groups which we called regular and irregular. For regular actions the static field equations allow a simple gauge choice such that with suitable boundary conditions a nonlinear and singular boundary value problem for a number of functions of the radial variable r results which is numerically difficult to solve but feasible (see, for example [20,21]). In the irregular case there appears to be no simple gauge choice to eliminate some dependent variables and the boundary value problem becomes degenerate and thus numerically quite unstable.

The purpose of this paper is two-fold. On the one hand we analyze carefully the possible actions of a symmetry group on both the principal and associated bundles and extend the classification of the conjugacy classes of action by automorphisms on principal bundles by Brodbeck [22] to those on associated bundles. This construction works for quite general symmetry group actions, structure groups and representations of the structure group on vector bundles subject to only very mild restrictions on the orbit structure. The main result is Theorem 1 which also allows us to make a fairly natural gauge choice in the general and in particular also the spherically symmetric case.

On the other hand we use this general result to classify the possible static spherically symmetric field equations of a general Einstein–Yang–Mills–Higgs (EYMH) system for arbitrary compact semisimple gauge groups, arbitrary symmetry group actions, and arbitrary representations defining the associated bundle whose sections are the generalized Higgs fields. This classification also applies, of course, to the Yang–Mills–Higgs theory where some early work [23] was done before the study of the EYM equations started. When generalized Higgs fields are included many results of the representation theory of compact Lie groups can be used.

We work with theories for which the Lagrange density is of the form $\mathcal{L}\sqrt{|g|}$ with

$$\mathcal{L} = \varkappa R - 2\Lambda - k(F_{\alpha\beta}, F^{\alpha\beta}) - h(D_\alpha\Phi, D^\alpha\Phi) - \mathcal{W}(h(\Phi, \Phi)), \tag{1}$$

where R is the scalar curvature of the metric $g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$, k is an ad-invariant positive definite inner product on the Lie algebra $\mathfrak{g}_o$ of the gauge group G_o, and h a (Hermitian) inner product on a (in general complex) vector space V, invariant under the action $\rho : G_o \to GL(V)$ and $\mathcal{W}$ is a scalar function of its argument serving as a potential. $D_\alpha\Phi$ denotes the gauge covariant derivative of the Higgs field Φ which depends on the metric, the gauge potential and ρ. (Also, $\varkappa = c^4/8\pi G$ with G being Newton's constant, Λ is the cosmological constant while coupling constants for the Yang–Mills and Higgs fields can be absorbed into the definitions of the inner products k and h.)

Part of our assumptions is that both the gauge field and the Higgs field are invariant under the appropriate actions of the symmetry group on the principal and the associated bundle, respectively. We then find that nontrivial spherically symmetric Higgs fields do not exist for certain representations ρ like a two-dimensional irreducible spinor representation, for example. This does not exclude the possibility of such a (noninvariant) Higgs field contributing to a spherically symmetric stress-energy tensor, however, and thus being compatible with a spherically symmetric space-time metric. But we are not aware of any reasonable definition of symmetry in which such Higgs fields themselves could be regarded as spherically symmetric.

The paper is organized as follows. In Sections 2 and 3 we recall the definition of automorphisms and automorphism groups of associated bundles and establish some notation. The general classification of invariant gauge and Higgs fields under any symmetry group action is obtained in Section 4. In Section 5 we specialize to the symmetry group $K = SU(2)$ and derive all field equations. We then verify that a consistent set of first and second order ordinary differential equations in the radial variable is obtained, subject to a set of constraint equations that need be satisfied only at one regular point and are then 'conserved'. This result is true whether or not the action of K is regular or not, but as in the pure EYM case the gauge choice is simple only in the regular case. In the final subsection we use results from [16,18] to cast the field equations into a fairly explicit form from which a numerical algorithm can be derived. We do not, in this paper, attempt to find new numerical solutions nor do we analyze what kind of boundary conditions are implied by regularity assumptions on the solution at singular points like the center ($r = 0$) or at a black hole horizon. In the general case (arbitrary compact gauge group and arbitrary representation) this is likely to lead to a considerable number of nontrivial algebraic problems as one can guess from the experience with the EYM fields. It is, however, a necessary first step for a numerical exploration of the solution set.

2. Associated bundles and their automorphisms

Let $P = (P, \pi, M, G_o, R)$ be a principal bundle over a manifold M with projection π, structure group G_o and right action R of G_o on P, $R : P \times G_o \to P : (p, g) \mapsto R_g p$.

Given another manifold V and a left action $\rho : G_o \times V \to V$ the *associated bundle* $E = (E, P, \pi_E, M, G_o, R, \rho, V)$ is defined as the set of equivalence classes $[p, v]$ of elements $(p, v) \in P \times V$ with respect to the relation:

$$(p', v') \sim (p, v) \Leftrightarrow p' = R_g p \quad \text{and} \quad v' = \rho_{g^{-1}} v \text{ for some } g \in G_o.$$

We denote the fibers of P and E over $x \in M$ by $P_x = \pi^{-1}(x)$ and by $E_x = \pi_E^{-1}(x)$, respectively.

We will, in general, assume that the action ρ is *effective*, i.e. that $\rho_g v = v\,\forall\, v \in V \Rightarrow g = e$, the identity of G_o. It then follows that $p, q \in P_x$, $[p, v] = [q, v]\,\forall\, v \in V \Rightarrow p = q$. Let $\hat{\pi} : P \times V \to E : (p, v) \mapsto [p, v]$ denote the canonical projection. Then $\hat{\pi}_p : V \to \pi_E^{-1}(\pi(p)) : v \mapsto [p, v]$ is an isomorphism of V onto $E_{\pi(p)}$ (a diffeomorphism in general, a vector space isomorphism if V is a vector space). We note that $\hat{\pi}_p^{-1}([p, v]) = v$ and also $\hat{\pi}_{R_g p} = \hat{\pi}_p \circ \rho_g$.

It is well known (e.g. [5]) that there is a one-to-one correspondence between the set $C\pi_E$ of sections Φ of E and equivariant maps $\tilde{\Phi} : P \to V$ (i.e. maps satisfying $\tilde{\Phi} \circ R_g = \rho_{g^{-1}} \circ \tilde{\Phi} \,\forall\, g \in G_o$) given by $\Phi \mapsto \tilde{\Phi}$ with $\tilde{\Phi}(p) = \hat{\pi}_p^{-1} \circ \Phi \circ \pi(p)$ and $\tilde{\Phi} \mapsto \Phi$ with $\Phi(x) = \hat{\pi}_p \circ \tilde{\Phi}(p) = [p, \tilde{\Phi}(p)]$ for any $p \in P_x$ and $x \in M$.

An *automorphism of P* is a diffeomorphism ψ of P onto itself such that $\pi \circ \psi = \bar{\psi} \circ \pi$ and $\psi \circ R_g = R_g \circ \psi \,\forall\, g \in G_o$ where $\bar{\psi}$ is an induced diffeomorphism of M onto itself. An *automorphism of E* is a bundle isomorphism $(\chi, \bar{\psi})$ of E, i.e. satisfying $\pi_E \circ \chi = \bar{\psi} \circ \pi_E$, and inducing an isomorphism of E_x onto $E_{\bar{\psi}(x)}$ for any $x \in M$ that is of the form $\chi = \hat{\pi}_q \circ \hat{\pi}_p^{-1} : E_x \to E_{\bar{\psi}(x)} : [p, v] \mapsto [q, v]$ for a certain $p \in P_x := \pi^{-1}(x)$ and $q \in P_{\bar{\psi}(x)}$ (cf. [24, p. 55]). Given an automorphism ψ of P there is, however, a natural way to induce a related automorphism ψ^E of an associated bundle E, namely by choosing $q = \psi(p)$ so that

$$\psi^E = \hat{\pi}_{\psi(p)} \circ \hat{\pi}_p^{-1} : E_x \to E_{\bar{\psi}(x)} : [p, v] \mapsto [\psi(p), v] \text{ for any } p \in P_x. \tag{2}$$

In this case *a section $\Phi \in C\pi_E$ is invariant under an automorphism ψ_E of E, i.e. satisfies $\Phi \circ \bar{\psi} = \psi^E \circ \Phi$ iff the corresponding equivariant map $\tilde{\Phi} : P \to V$ is invariant in the sense of satisfying $\tilde{\Phi} \circ \psi = \tilde{\Phi}$* (see [25], for example). Every automorphism of E is induced by one of P in this way, provided that the action ρ defining E is effective.

We now describe both ψ and ψ^E with respect to a local trivialization $U \times G_o$ of P where U is an open set of M. Given a local section $\sigma : U \to P$ define a local trivialization of P by

$$\tau : U \times G_o \to \pi^{-1}(U) : (x, g) \mapsto R_g \sigma(x) \tag{3}$$

and let

$$\tau_E : U \times V \to \pi_E^{-1}(U) : (x, v) \mapsto [\sigma(x), v] \tag{4}$$

be the associated local trivialization of E. With respect to this trivilization a section Φ of E can always be written as

$$\Phi(x) = \tau_E(x, \phi(x)) = [\sigma(x), \phi(x)], \quad x \in U \tag{5}$$

for some map $\phi : U \to V$.

The corresponding equivariant map $\tilde{\Phi}$ then satisfies $\tilde{\Phi} \circ \tau(x, g) = \rho_{g^{-1}} \phi(x)$ and under a gauge change a Higgs field Φ transforms like $\phi(x) \mapsto \check{\phi}(x) = \rho_{\gamma(x)^{-1}} \phi(x)$.

In such a local chart we can describe the automorphism ψ in the form $\psi(\sigma(x)) = R_{\hat{\psi}(x)} \sigma(\bar{\psi}(x))$ where $\hat{\psi} : U \to G_o$. Then $\psi(\tau(x, g)) = \tau(\bar{\psi}(x), \hat{\psi}(x) g)$. Similarly, the map $\psi^E([\sigma(x), v]) := [R_{\psi^E(x)} \sigma(\bar{\psi}(x)), v] = [\sigma(\bar{\psi}(x)), \rho_{\psi^E(x)} v]$ is locally of the form $\psi^E \circ \tau_E(x, v) \tau_E(\bar{\psi}(x), \rho_{\hat{\psi}(x)} v)$. Under a local gauge transformation $\sigma_2(x) = R_{\gamma(x)} \sigma_1(x)$ the functions $\hat{\psi}_1$ changes into $\hat{\psi}_2 = \gamma(\bar{\psi}_1(x))^{-1} \hat{\psi}_1(x) \gamma(x)$.

3. Symmetry group acting on *P* and *E*

We will be interested in groups of automorphisms of P and of E that cover the same diffeomorphisms of M and want to explore just how many independent choices can be made to describe such actions completely. For classical relativistic field theories we would expect to have an isometry group of a Lorentzian space-time manifold M and, if there are gauge fields and Higgs fields present, we would expect this group to lift to act by automorphisms on the bundles P and E so that all physical fields are invariant under this symmetry group action. (This is to some extent even implied if Einstein's equations hold because then, if the metric is invariant, the whole stress-energy tensor will have to be invariant too which imposes strong constraints on the gauge and Higgs fields although it does not imply that they are invariant under the symmetries.)

For the remainder of this article we will assume that that the associated bundle $E = P \times_\rho V$ is a *vector bundle*. That is V will be taken to be a finite-dimensional vector space and $\rho : G_o \to \mathrm{GL}(V)$ a *linear representation* of G_o on V. We will also assume that there is a positive definite Hermitian inner product $h : V \times V \to \mathbb{R}$ on V that is invariant under the action of G_o. In other words $h(\rho_g v, \rho_g w) = h(v, w)$ for all $g \in G_o$ and $v, w \in V$. We note that if G_o is compact, then there will always exist such an inner-product.

We call a principal bundle $P = P(M, G_o)$ on which a Lie group K acts effectively on the left

$$\psi : K \times P \longrightarrow P : (a, p) \longmapsto \psi_a p \tag{6}$$

by principal bundle automorphisms a *K-symmetric principal bundle*. Let $\bar{\psi} : K \times M \longrightarrow M : (a, x) \longmapsto \bar{\psi}_a x$ denote the left action of K induced on M via projections of ψ_a. As discussed in the previous section, the action (6) induces a *natural* left action of K on E by bundle automorphisms which is given by

$$\psi^E : K \times E \longrightarrow E : (a, [p, v]) \mapsto [\psi_a(p), v] \tag{7}$$

and in local coordinates by

$$\psi_a(x, g) = (\bar{\psi}_a x, \hat{\psi}(a, x) g) \quad \text{and} \quad \psi_a^E(x, v) = (\bar{\psi}_a x, \rho_{\hat{\psi}(a,x)} v). \tag{8}$$

The symmetry group action is therefore determined (in a given gauge) by two maps $\bar{\psi} : K \times U \to U$ and $\hat{\psi} : K \times U \to G_o$. The fact that K acts on the left implies $\bar{\psi}(ab, x) = \bar{\psi}(a, \bar{\psi}(b, x))$, $\hat{\psi}(e_K, x) = e_{G_o}$, and $\hat{\psi}(ab, x) = \hat{\psi}(a, \bar{\psi}_b x)\hat{\psi}(b, x)$ for all $a, b \in K$ and $x \in U$.

It is easily seen that ψ_E induces a right action ψ_E^* on the set $C\pi_E$ of sections of E by

$$\psi_a^{E*}(\Phi) := \psi_{a^{-1}}^E \circ \Phi \circ \bar{\psi}_a \quad \forall\, a \in K. \tag{9}$$

Therefore a section Φ is called *invariant under the action of K* if

$$\psi_a^{E*}(\Phi) = \Phi \quad \forall\, a \in K.$$

The invariant Hermitian inner-product h can be used to induce a Hermitian inner product on the vector bundle E. If $\sigma : U \subset M \to P$ is a local section and $\Phi, \Psi \in C\pi_E$ are two sections with local representatives $\Phi^\sigma : U \to V$ and $\Psi^\sigma : U \to V$, respectively, so that

$$\Phi(x) = [\sigma(x), \Phi^\sigma(x)] \quad \text{and} \quad \Psi(x) = [\sigma(x), \Phi^\sigma(x)] \quad \text{for all } x \in U,$$

then the Hermitian metric h on E is defined by the formula:

$$h(\Phi, \Phi) := h(\Phi^\sigma, \Psi^\sigma) \quad \text{for all } x \in U. \tag{10}$$

The G_o-invariance of h guarantees that this local formula defines a global Hermitian metric.

4. Classifying invariant Higgs fields

As in the previous section we assume that $P = P(M, G_o)$ is a K-symmetric bundle and that K acts on the vector bundle $E = P \times_\rho V$ according to the natural action (7). Also, for the remainder of this article we will assume that the symmetry group K is *compact*. Once we know this, then we know that there exists an open dense subset $U \subset M$ such that U is, at least locally, regularly foliated by orbits of K under the action $\bar{\psi}$ on M. Fixing a point $x_o \in U$ and letting K_o be the isotropy group of x_o, we then have that locally $U \approx U/K \times K/K_o$. This shows that we can, with a minor loss of generality, assume that $M = \tilde{M} \times K/K_o$ with the $\bar{\psi}$ action given by

$$\bar{\psi} : K \times (\tilde{M} \times K/K_o) \longrightarrow \tilde{M} \times K/K_o : (a, (x, kK_o)) \longmapsto (x, akK_o).$$

In [22] it is established that the K-symmetric principal bundles over base manifolds of the form $\tilde{M} \times K/K_o$ can be classified by a homomorphism $\lambda : K_o \to G_o$ and a principal bundle $\tilde{Q}$ over $\tilde{M}$ with structure group $Z := \mathrm{Cent}(\lambda(K_o)) \subset G_o$. The classifying bundle $\tilde{Q}$ is constructed as follows. Let $P|_{\tilde{M}}$ be the portion of P over the submanifold $\tilde{M} \cong \tilde{M} \times \{eK_o\}$ (i.e. $P|_{\tilde{M}} := \{p \in P : \pi(p) \in \tilde{M} \times eK_o\}$). Then $\tilde{M} \cong \tilde{M} \times \{eK_o\}$ is a fixed point set for the action $\bar{\psi}$ and hence each fiber of $P|_{\tilde{M}}$ is mapped onto itself by the action of K_o on P. This induces a map $\mu : P|_{\tilde{M}} \times K_o \to G_o : (p, h) \mapsto \mu_p(h)$ where $\mu_p(h)$ is the unique element of G_o satisfying $\psi_h(p) = R_{\mu_p h}(p)$. For each $p \in P|_{\tilde{M}}$, $\mu_p : K_o \to G_o$ defines a group homomorphism. Moreover, if $p, q \in P|_{\tilde{M}}$ are in the same fiber then the homomorphisms μ_p and μ_q belong to the same conjugacy class. Next, fix $p_o \in P|_{\tilde{M}}$ and let $\lambda := \mu_{p_o}$. Then $\tilde{Q}$ is defined by

$$\tilde{Q}(\tilde{M}, Z) := \{p \in P|_{\tilde{M}} : \mu_p = \lambda\} \tag{11}$$

and it can be shown that $\tilde{Q}$ is a principal bundle over $\tilde{M}$ with structure group Z. Thus each K-symmetric principal bundle P with base manifold $M = \tilde{M} \times K/K_o$ determines a Z-bundle $\tilde{Q}$ over $\tilde{M}$ and a homomorphism $\lambda : K_o \to G_o$. Conversely, given $(\tilde{Q}, \lambda)$ it is possible to construct a bundle isomorphic to P. We describe the construction below because it produces a principal bundle $\hat{P}$ isomorphic to P on which the K action is made as simple as possible. This makes it easy to identify the K-invariant Higgs fields.

Let $\check{P} := \tilde{Q} \times K$ be the product bundle with base $\tilde{M} \times K/K_o$ and gauge group $\check{G}_o := Z \times K_o$ which acts on $\check{P}$ via $R_{(z,h)}(q, k) = (q, k) \cdot (z, h) := (R_z(q), kh)$ for all $(z, h) \in Z \times K_o$. The projection $\pi_{\check{P}} : \check{P} \to \tilde{M} \times K/K_o$ is given by $\pi_{\check{P}}(q, k) = (\pi_{\tilde{Q}}(q), kK_o)$ where $\pi_{\tilde{Q}} : \tilde{Q} \to \tilde{M}$ is the principal bundle projection map. Clearly, $\psi' : K \times \check{P} \to \check{P} : (k_1, (q, k)) \mapsto (q, k_1k)$ is a left action of K on $\check{P}$ by bundle automorphisms. We let ρ_λ be the homomorphism defined by $\rho_\lambda : \check{G}_o \to G_o : (z, h) \mapsto z\lambda(h)$, and let $\check{G}_o$ act on G_o via $\check{G}_o \times G_o \to G_o : (g', g) \mapsto \rho_\lambda(g')g$. This allows us to define the associated bundle $\hat{P} := \check{P} \times_{\rho_\lambda} G_o$. It can be verified that $\hat{P}$ is a principal bundle with base $\tilde{M} \times K/K_o$ and structure group G_o with the right action of G_o given by $\hat{R}_{g_1}([(q, k), g]) = [(q, k), g] \cdot g_1 := [(q, k), gg_1]g_1 \in G_o$. The bundle projection map $\pi_{\hat{P}} : \hat{P} \to \tilde{M} \times K/K_o$ is given by $\pi_{\hat{P}}([(q, k), g]) := \pi_{\check{P}}(q, k)$. As indicated above, the importance of $\hat{P}$ is that it is isomorphic to P with the isomorphism defined by $\hat{P} \to P : [(q, k), g] \mapsto \psi_k R_g q$. This defines a K and G_o equivariant bundle isomorphism that induces the identity on the common base $\tilde{M} \times K/K_o$ of the two bundles $\hat{P}$ and P. We also note that left action of K on $\hat{P}$ is given simply by

$$\hat{\psi} : K \times \hat{P} \longrightarrow \hat{P} : (k_1, [(q, k), g]) \longmapsto [(q, k_1k), g]. \tag{12}$$

Now that the bundle $\hat{P}$ has been defined, we can use it to classify the invariant Higgs fields. Consider the associated vector bundle $\hat{E} := \hat{P} \times_\rho V$ with projection $\pi_{\hat{E}} : \hat{E} \to \tilde{M} \times K/M$. The points of $\hat{E}$ are the equivalence classes $[[(q, k), g], v]$ where $[(q, k), g]$ is a point in $\hat{P}$ and v is a vector in V. We note that the natural left K-action $\psi^{\hat{E}} : K \times \hat{E} \to \hat{E}$ is given by

$$\psi^{\hat{E}} : K \times \hat{E} \longrightarrow \hat{E} : (k_1, [[(q, k), g], v]) \longmapsto \psi^{\hat{E}}_{k_1}([[(q, k), g], v]) := [[(q, k_1k), g], v].$$

This follows from (12) and (7). Let $\Phi : \tilde{M} \times K/K_o \to \hat{E}$ be a section of $\hat{E}$ that is K-invariant in the sense of (9). In other words, Φ is a K-invariant Higgs field. From Section 2, we know that

Φ is equivalent to a G_o-equivariant map $\tilde{\Phi} : \hat{P} \to V$ which satisfies $\tilde{\Phi} \circ \hat{\psi}_k = \tilde{\Phi}$ for all $k \in K$. Letting $\pi_{\check{P}\times G_o}$ denote the projection $\pi_{\check{P}\times G_o} : \check{P} \times G_o \to \hat{P} : (p, g) \mapsto [p, g]$, we then find that the diagram

$$\begin{array}{ccccc} \check{P} \times G_o & \xrightarrow{\pi_{\check{P}\times G_o}} & \hat{P} & \xrightarrow{\tilde{\Phi}} & V \\ \downarrow {\scriptstyle \mathrm{pr}_1} & & \downarrow {\scriptstyle \pi_{\hat{P}}} & & \\ \check{P} & \xrightarrow[\pi_{\check{P}}]{} & \widetilde{M} \times K/K_o & & \end{array}$$

commutes. It is not difficult to see from this diagram that $\pi_{\check{P}\times G_o}$ defines a K-equivariant principal bundle homomorphism between $\check{P} \times G_o$ and $\hat{P}$ that induces the identity homomorphism on G_o. Moreover, from the surjectivity of $\pi_{\check{P}\times G_o}$, it is clear that the equivariant map corresponding to the Higgs field $\hat{\Phi}$ is completely determined by the map $\check{\Phi} := \hat{\Phi} \circ \pi_{\check{P}\times G_o} : \check{P} \times G_o \to V$. Since $\pi_{\check{P}\times G_o}$ is both G_o and K equivariant, it follows that $\check{\Phi}$ is K and G_o equivariant. That is $\check{\Phi}$ satisfies

$$\check{R}_g \circ \check{\Phi} = \rho(g^{-1}) \circ \check{\Phi} \quad \text{for all } g \in G_o \tag{13}$$

and

$$\check{\Phi} \circ \psi'_k = \check{\Phi} \quad \text{for all } k \in K. \tag{14}$$

The map $\check{\Phi}$ possesses an additional invariance coming from the construction of the bundle $\hat{P}$. Letting ϕ_g denote the action

$$\phi_{g'} : \check{P} \times G_o \longrightarrow \check{P} \times G_o : ((q, k), g) \longmapsto (R_{g'}((q, k)), \rho_\lambda(g'^{-1})g) \quad \text{for } g' \in \check{G}_o$$

it follows from the definition of $\hat{P}$ as the associated bundle that $\pi_{\check{P}\times G_o} \circ \phi_{g'} = \pi_{\check{P}\times G_o}$ for all $g' \in \check{G}_o$. Consequently $\check{\Phi}$ satisfies

$$\check{\Phi} \circ \phi_{g'} = \check{\Phi} \quad \text{for all } g' \in \check{G}_o. \tag{15}$$

From (13) we have that $\check{\Phi}((q, k), g) = \rho_{g^{-1}}\check{\Phi}((q, k), e)$ while (14) shows that $\check{\Phi}((q, k), g) = \check{\Phi}((q, e), g)$. Combining these two results yields

$$\check{\Phi}((q, k), g) = \rho_{g^{-1}}\check{\Phi}((q, e), e). \tag{16}$$

We also have from (15) that

$$\check{\Phi}((q, k), g) = \rho_{g^{-1}}\check{\Phi}((q \cdot z, kh), z^{-1}\lambda h^{-1}g). \tag{17}$$

Eqs. (16) and (17) imply that

$$\check{\Phi}((q \cdot z, e), e) = \rho_{z^{-1}}\rho_{\lambda(h^{-1})}\check{\Phi}((q, e), e) \tag{18}$$

for all $q \in \tilde{Q}$ and $z \in Z$, $h \in K_o$. Defining the map

$$\tilde{L} : \tilde{Q} \longrightarrow V : q \longmapsto \tilde{L}(q) := \check{\Phi}((q, e), e), \tag{19}$$

Eq. (16) shows that $\tilde{L}$ and the action ρ can be used to completely determine the invariant Higgs field via the relationship $\tilde{\Phi}([(q, k), g]) = \rho_{g^{-1}}\tilde{L}(q)$ for all $q \in \tilde{Q}$, $k \in K$, and $g \in G_o$. This can also be written as

$$\Phi \circ \pi_{\check{P}}(q, k) = [[(q, k), g], \rho_{g^{-1}}\tilde{L}(q)] \quad \text{for all } q \in \tilde{Q}, k \in K \text{ and } g \in G_o. \tag{20}$$

The map $\tilde{L}$ is not arbitrary but is in fact a section of the associate bundle $\tilde{Q} \times_{(\rho,Z)} V$. To see this set $h = e$ in (18) to get $\tilde{L} \circ R_z = \rho_{z^{-1}} \circ \tilde{L}$ for all $z \in Z$. This shows that $\tilde{L} : \tilde{Q} \to V$ is an equivariant map and hence by the discussion in Section 2, uniquely determines a section of the vector bundle $\tilde{Q} \times_{(\rho,Z)} V$. Setting $z = e$ in (18) shows that $\tilde{L}$ must satisfy also the additional condition

$$\rho_{\lambda(h)}\tilde{L}(q) = \tilde{L}(q) \quad \text{for all}\, q \in \tilde{Q} \,\text{and all}\, h \in K_o. \tag{21}$$

We summarize the above results in the following theorem.

Theorem 1. *Let $P(M, G_o)$ be a principal bundle and suppose that*

- (i) *K is a compact Lie group that acts on $P(M, G_o)$ on the left by principal bundle automorphisms*;
- (ii) *the base space M is diffeomorphic to $\tilde{M} \times K/K_o$ where $\tilde{M}$ is a smooth manifold, K_o is the isotropy subgroup of K of any point $x_o \in M$, and the induced action of K on M is given by $(k_1, (x, kK_o)) \to (x, k_1kK_o)$*;
- (iii) *the K-symmetric bundle $P(M, G_o)$ is classified by a homomorphism $\lambda : K_o \to G_o$ and a principal bundle $\tilde{Q}(\tilde{M}, Z)$ $(Z := \mathrm{Cent}(\lambda(K_o)) \subset G_o)$ in the sense of Brodbeck* (*see* (11) *and* [22]);
- (iv) *V is a vector space, ρ is a linear representation of G_o on V, and $E := P \times_\rho V$ is the associated vector bundle.*

Then the set of K-invariant sections of the vector bundle E is in one-to-one correspondence with the set of maps $\tilde{L} : \tilde{Q} \to V$ satisfying

$$\tilde{L} \circ R_z = \rho_{z^{-1}} \circ \tilde{L} \quad \textit{and} \quad \rho_{\lambda(h)} \circ \tilde{L} = \tilde{L}$$

for all $z \in Z$ and $h \in K_o$.

In order to derive the EYMH equations we need to have explicit local formulae for the invariant fields. This means fixing a gauge. To fix the gauge, let $\tilde{\sigma} : \tilde{U} \subset \tilde{M} \to \tilde{Q}$ and $\hat{\sigma} : \hat{U} \subset K/K_o \to K$ be two local sections. Then, if $\iota_e : \check{P} \to \check{P} \times G_o : p \mapsto (p, e)$, the two local sections $\tilde{\sigma}$ and $\hat{\sigma}$ can be used to define local section (i.e. a gauge) of $\hat{P}$ by

$$\sigma : \tilde{U} \times \hat{U} \subset \tilde{M} \times K/K_o \to \hat{P}, \qquad \sigma := \pi_{\check{P} \times G_o} \circ \iota_e \circ \tilde{\sigma} \times \hat{\sigma}. \tag{22}$$

In [22] it is shown that the set of K-invariant connection forms ω on P is in one-to-one correspondence with the set of pairs $(\tilde{\omega}, \tilde{\Lambda})$ where $\tilde{\omega}$ is a connection form on $\tilde{Q}$ and $\tilde{\Lambda}$ is a map $\tilde{\Lambda} : \tilde{Q} \longrightarrow \mathfrak{gl}(\mathfrak{k}, \mathfrak{g})$ that satisfies

$$\tilde{\Lambda}(q) \circ \mathrm{Ad}_h = \mathrm{Ad}_{\lambda(h)}\tilde{\Lambda}(q) \quad \text{and} \quad \tilde{\Lambda}(q) \cdot \xi = \lambda'(\xi) \quad (\lambda' := T_e\lambda) \tag{23}$$

for all $q \in \tilde{Q}$, $h \in K_o$, and $\xi \in \mathfrak{k}_o$. Moreover, it is shown in [22] that in the gauge (22) the gauge potential $A^\sigma := \sigma^*\omega$ can be written as $A^\sigma = \tilde{A}^\sigma + \Lambda^\sigma\hat{\sigma}^*\theta^K$ where $\tilde{A}^\sigma := \tilde{\sigma}^*\tilde{\omega}$, $\Lambda^\sigma := \tilde{\Lambda} \circ \tilde{\sigma}$, and $\theta^K_h := T_h\ell_{h^{-1}}$ (ℓ_h being the left translation by h) is the Maurer–Cartan form on K. This takes care of the local formula for the gauge potential.

We now consider the Higgs field. For $(y, kK_o) \in \tilde{U} \times \hat{U}$ we have $\sigma(y, kK_o) = [(\tilde{\sigma}(y), \hat{\sigma}(kK_o)), e]$ and so it follows from (20) that $\Phi(y, kK_o) = [\sigma(y, kK_o), \tilde{L} \circ \tilde{\sigma}(y)]$ for all $(y, kK_o) \in \tilde{U} \times \hat{U}$. This shows that in the gauge (22) (see (5)) the Higgs field is given by

$$\phi^\sigma(y, kK_o) = L^\sigma(y) := \tilde{L} \circ \tilde{\sigma}(y), \quad (y, kK_o) \in \tilde{U} \times \hat{U}.$$

In view of (21), L^σ must satisfy $\rho_{\lambda(h)}L^\sigma(y) = L^\sigma(y)\,\forall\, h \in K_o, \forall\, y \in \tilde{U}$. which becomes infinitesimally,

$$\rho_{\lambda'(\xi)}L^\sigma(y) = 0 \quad \forall\, \xi \in \mathfrak{k}_o, \forall\, y \in \tilde{U}, \tag{24}$$

where $\rho : \mathfrak{g}_o \longrightarrow \mathfrak{gl}(V)$ is the Lie algebra representation of $\mathfrak{g}_o$ on V induced from the group representation ρ.

Another important fact that will be needed for the analysis of the EYMH equations is that invariant Higgs fields produce an invariant stress-energy tensor. The Higgs field contribution to the stress-energy tensor is made from the two following combinations of the Higgs field: $h(\Phi, \Phi)$ and $h(D_X\Phi, D_Y\Phi)$ where D_X is the covariant derivative on E and h is the Hermitian metric on E (see (10)).

Eqs. (8) and (10), and the G_o-invariance of h imply that

$$h(\psi_k^E\Psi, \psi_k^E\Phi) = h(\Psi, \Phi) \tag{25}$$

for all $(\Psi, \Phi) \in C\pi_E \times C\pi_E$ and $k \in K$. Therefore any K-invariant section Φ satisfies

$$\bar{\psi}_k^*(h(\Phi, \Phi)) = h(\Phi, \Phi) \quad \text{for all}\, k \in K. \tag{26}$$

This shows that $h(\Phi, \Phi)$ defines a K-invariant function on M.

If we assume that the connection on P is K-invariant, then a straightforward computation using the invariance of the connection form ω (i.e. $\psi_k\omega = \omega$ for all $k \in K$) and the definition of the covariant derivative (see [24, Chapter III, Section 1]) shows that if Φ is a K-invariant section of E then

$$(D_{\bar{\psi}_{k*}X}\Phi)(\bar{\psi}_k(x)) = \psi_k^E(D_X\Phi(x)) \tag{27}$$

for all $x \in M$, $k \in K$, and all vector fields X on M.

Now suppose that X and Y are two vector fields on M. Then

$$H(X, Y) := h(D_X\Phi, D_Y\Phi) \tag{28}$$

defines a rank 2 covariant tensor and

$$(\bar{\psi}_k^*H)(X, Y) = h(D_{\bar{\psi}_{k*}X}\Phi \circ \bar{\psi}_k, D_{\bar{\psi}_{k*}Y}\Phi \circ \bar{\psi}_k) \overset{(27)}{=} h(\psi_k^E D_X\Phi, \psi_k^E D_Y\Phi) \overset{(25)}{=} h(D_X\Phi, D_Y\Phi)$$

shows that H is a K-invariant tensor field on M, that is

$$\bar{\psi}_k^*H = H \quad \text{for all}\, k \in K. \tag{29}$$

5. Spherically symmetric field equations

5.1. Field equations in general

Variation of the action corresponding to the Lagrangian (1) with respect to the metric, the gauge potential components and the (real and imaginary) components of a Higgs field yields the field equations:

$$\varkappa(R_{\alpha\beta} - \tfrac{1}{2}Rg_{\alpha\beta}) + \Lambda g_{\alpha\beta} = T_{\alpha\beta}, \tag{30}$$

$$k(A, D^\mu F_{\mu\alpha}) = 2\,\mathbf{Re}\,h(\rho_A\Phi, D_\alpha\Phi) \quad \forall\, A \in \mathfrak{g}_o, \tag{31}$$

$$D^\mu D_\mu \Phi - 2\mathcal{W}' \Phi = 0, \tag{32}$$

where

$$T_{\alpha\beta} = k(F_{\alpha\mu}, F^\mu_\beta) - \tfrac{1}{4}k(F_{\lambda\mu}, F^{\lambda\mu})g_{\alpha\beta} + h(D_{(\alpha}\Phi, D_{\beta)}\Phi) \\ - \tfrac{1}{2}h(D_\lambda\Phi, D^\lambda\Phi)g_{\alpha\beta} - \tfrac{1}{2}\mathcal{W}g_{\alpha\beta}. \tag{33}$$

Eq. (31) can be written in the form

$$D^\mu F_{\mu\alpha} = \tilde{\rho}(\Phi, D_\alpha\Phi), \tag{34}$$

where $\tilde{\rho} : V \times V \to \mathfrak{g}_o$ is defined by

$$k(A, \tilde{\rho}(x, y)) = h(\rho_A x, y) + h(y, \rho_A x) = h(y, \rho_A x) - h(x, \rho_A y) \quad \forall x, y \in V, \forall A \in \mathfrak{g}_o \tag{35}$$

(the second formula being true because ρ_A is a anti-Hermitian operator on V). It then follows from the invariance properties of k and h:

$$k([A, B], C) = k(A, [B, C]) \quad \forall A, B, C \in \mathfrak{g}_o, \tag{36}$$

$$h(\rho_A x, y) + h(x, \rho_A y) = 0 \quad \forall A \in \mathfrak{g}_o, \forall x, y \in V, \tag{37}$$

that the map $\tilde{\rho}$ satisfies

$$\tilde{\rho}(x, y) = -\tilde{\rho}(y, x), \tag{38}$$

$$k([A, B], \tilde{\rho}(x, y)) = k(A, \tilde{\rho}(\rho_B x, y)) - k(B, \tilde{\rho}(\rho_A x, y)). \tag{39}$$

If $k_{\Gamma\Delta}$ and h_{IJ} are the components of k and h with respect to bases $\{\mathbf{e}_\Gamma\}$ of $\mathfrak{g}_o$ and $\{\mathbf{E}_I\}$ of V, respectively, then $\tilde{\rho}$ can be given by

$$\tilde{\rho}^\Gamma_{IJ} := -2k^{\Gamma\Sigma}\rho^K_{\Sigma[I}h_{J]K}, \tag{40}$$

where $(k^{\Gamma\Delta})$ is the inverse matrix to $(k_{\Gamma\Delta})$ and $\rho_{\mathbf{e}_\Gamma}(\mathbf{E}_J) = \mathbf{E}_K\rho^K_{\Gamma J}$.

In the special case where ρ is the adjoint representation, the map $\tilde{\rho}$ is given by the negative of the Lie bracket, i.e. $\tilde{\rho}(A, B) = -[A, B]$.

5.2. *Spherically symmetric EYMH fields*

The spherically symmetric space-time metric can be given in a Schwarzschild-like coordinate system by

$$g = -NS^2\,\mathrm{d}t^2 + N^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{41}$$

where N and S are functions of r and t, in general, and of r only in the static case. The function N is related to the mass function $m(r, t)$ by $N = 1 - 2m/r - \Lambda/(3\varkappa)r^2$. We assume that the space-time M is diffeomorphic to $\tilde{M} \times S^2$ where $\tilde{M}$ is the 'r-t' manifold and S^2 the orbits of the symmetry group action.

The Yang–Mills potential for the gauge group $G = SU(2)$ has often been given in the so-called Witten form [26] which is, however, not easily generalized to other gauge groups. Potentials for general compact gauge groups (in the EYM case) have first been discussed by Bartnik [27] and by Brodbeck and Straumann [15]. They show that the gauge potential can be given in the form:

$$A = NS\mathcal{A}\,\mathrm{d}t + \mathcal{B}\,\mathrm{d}r + \Lambda_1\,\mathrm{d}\theta + (\Lambda_2\,\sin\theta + \Lambda_3\,\cos\theta)\,\mathrm{d}\varphi. \tag{42}$$

If we choose the symmetry group to be $K = SU(2)$ whose action on space-time has as isotropy subgroup $K_o = U(1)$ so that $K/K_o \simeq S^2$ then Λ is a map from $\tilde{M}$ into the space of linear maps from $\mathfrak{k}$ to $\mathfrak{g}_o$ subject to (23) which implies

$$[\Lambda_2, \Lambda_3] = \Lambda_1 \quad \text{and} \quad [\Lambda_3, \Lambda_1] = \Lambda_2, \tag{43}$$

where $\Lambda_k = \Lambda(\tau_k)$, $\{\tau_k : k = 1, 2, 3\}$ being the standard basis of $\mathfrak{su}(2)$ with τ_3 spanning $\mathfrak{k}_o$. So $\Lambda_3 = \lambda'(\tau_3) \in \mathfrak{g}_o$ is a constant vector characterizing the embedding of $SU(2)$ in G and thus the conjugacy class of the $SU(2)$-action on P. Also $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{g}_o$-valued functions on $\tilde{M}$ which, moreover, commute with Λ_3. They give the “electric” part of the Yang–Mills potential. One can choose a temporal gauge so that $\mathcal{B} = 0$, and since one is mostly interested in the noncommuting aspects of the Yang–Mills field the component $\mathcal{A}$ is often assumed to be zero, as we will also do from now on.

The static spherically symmetric field equations for the full EYMH system can now be written in a form just slightly more general than those derived in [16,18]. We need to observe that locally invariant Higgs fields are described by V-valued functions of r, i.e. maps $r \in \tilde{U} \subset \mathbb{R} \to V$, since here $H = U(1)$, subject to the condition (24) which becomes

$$\rho_{\Lambda_3}\Phi(r) = 0. \tag{44}$$

The Yang–Mills equations then become

$$r^2 S^{-1}(NS\Lambda_1')' - [\Lambda_2, \hat{F}] = r^2\tilde{\rho}(\Phi, \rho_{\Lambda_1}\Phi), \tag{45}$$

$$r^2 S^{-1}(NS\Lambda_2')' + [\Lambda_1, \hat{F}] = r^2\tilde{\rho}(\Phi, \rho_{\Lambda_2}\Phi), \tag{46}$$

$$[\Lambda_1', \Lambda_1] + [\Lambda_2', \Lambda_2] = r^2\tilde{\rho}(\Phi, \Phi'), \tag{47}$$

where

$$\hat{F} := [\Lambda_1, \Lambda_2] - \Lambda_3. \tag{48}$$

The Higgs equation takes the form

$$S^{-1}(r^2 NS\Phi')' + (\rho_{\Lambda_1}\rho_{\Lambda_1} + \rho_{\Lambda_2}\rho_{\Lambda_2})\Phi + \mathcal{W}'\Phi = 0. \tag{49}$$

To derive the expression for the stress-energy tensor repeated use must be made of (43) and (44), the representation property, $\rho_{[X,Y]} = \rho_X\rho_Y - \rho_Y\rho_X$ for $X, Y \in \mathfrak{g}_o$, as well as the assumption that ρ_X is anti-Hermitian on V. We find from these relations that

$$[\Lambda_3, \hat{F}] = 0, \qquad k(\Lambda_1', \hat{F}) = k(\Lambda_2', \hat{F}) = 0, \tag{50}$$

$$k(\Lambda_1, \Lambda_2) = k(\Lambda_1', \Lambda_2') = k(\Lambda_2, \Lambda_2) - k(\Lambda_1, \Lambda_1) = k(\Lambda_2', \Lambda_2') - k(\Lambda_1', \Lambda_1') = 0, \tag{51}$$

$$h(\rho_{\Lambda_2}\Phi, \rho_{\Lambda_2}\Phi) = h(\rho_{\Lambda_1}\Phi, \rho_{\Lambda_1}\Phi), \qquad h(\rho_{\Lambda_2}\Phi, \rho_{\Lambda_1}\Phi) = -h(\rho_{\Lambda_1}\Phi, \rho_{\Lambda_2}\Phi) \tag{52}$$

and then that $(T^\alpha_\beta) = \operatorname{diag}(-e, p_r, p_\theta, p_\theta)$ with

$$e = r^{-2}N\|\Lambda_1'\|^2 + \tfrac{1}{2}r^{-4}\|\hat{F}\|^2 + \tfrac{1}{2}N\|\Phi'\|^2 + r^{-2}\|\rho_{\Lambda_1}\Phi\|^2 + \tfrac{1}{2}\mathcal{W}, \tag{53}$$

$$p_r = r^{-2}N\|\Lambda_1'\|^2 - \tfrac{1}{2}r^{-4}\|\hat{F}\|^2 + \tfrac{1}{2}N\|\Phi'\|^2 - r^{-2}\|\rho_{\Lambda_1}\Phi\|^2 - \tfrac{1}{2}\mathcal{W}, \tag{54}$$

$$p_\theta = \tfrac{1}{2}r^{-4}\|\hat{F}\|^2 - \tfrac{1}{2}N\|\Phi'\|^2 - \tfrac{1}{2}\mathcal{W} \tag{55}$$

(where $\|\cdot\|$ is the norm induced by $k(,)$ or $h(,)$, respectively) so that the Einstein equations become

$$\varkappa m' = \tfrac{1}{2}r^2 e, \tag{56}$$

$$\varkappa S^{-1}S' = \tfrac{1}{2}rN^{-1}(e+p_r) = r^{-1}\|\Lambda_1'\|^2 + \tfrac{1}{2}r\|\Phi'\|^2. \tag{57}$$

5.3. Consistency of the spherically symmetric equations

Eq. (47) can be viewed as a constraint equation since Eqs. (45), (46) and (49) are second-order differential equations which when solved will fully determine Yang–Mills potential and the Higgs field. The next proposition shows that away from the singular points where $N(r)=0$, $S(r)=0$, or $r=0$ the constraint equation (47) is 'conserved', i.e. automatically satisfied if it is satisfied at one point and hence it is only a constraint on the initial data for the differential equations (45), (46) and (49). We suspect that as in the EYM case this will still hold for solutions defined about the singular point but we have not (yet) done an analysis of the differential equation near the singular points similar to that in [16,18].

Proposition 2. *Suppose* $\{N(r), S(r), \Lambda_1(r), \Lambda_2(r)\}$ *satisfy the Yang–Mills equations (45) and (46) and the Higgs equation (49) on an interval* $[r_1, r_2)$ $(r_1 > 0)$. *If neither* $N(r)$ *nor* $S(r)$ *vanish on the interval* $[r_1, r_2)$ *and if the constraint equation (47) holds at* $r = r_1$ *then it holds at all* $r \in [r_1, r_2)$.

Proof. Let

$$\gamma := [NS\Lambda_1', \Lambda_1] + [NS\Lambda_2', \Lambda_2] - \tilde{\rho}(\Phi, r^2 NS\Phi').$$

Differentiating γ and using Eqs. (45), (46) and (49) yields

$$\begin{aligned}\gamma' = {} & r^{-2}S([[\Lambda_2, \hat{F}], \Lambda_1] - [[\Lambda_1, \hat{F}], \Lambda_2]) + S([\tilde{\rho}(\Phi, \rho_{\Lambda_1}\Phi), \Lambda_1] \\ & + [\tilde{\rho}(\Phi, \rho_{\Lambda_2}\Phi), \Lambda_2] + \tilde{\rho}(\Phi, (\rho_{\Lambda_1}^2 + \rho_{\Lambda_2}^2)\Phi)).\end{aligned} \tag{58}$$

The Jacobi identity and Eq. (43) imply that

$$[[\Lambda_2, \hat{F}], \Lambda_1] - [[\Lambda_1, \hat{F}], \Lambda_2] = 0, \tag{59}$$

while for any $A \in \mathfrak{g}_o$ and $j = 1, 2$,

$$k(A, [\tilde{\rho}(\Phi, \rho_{\Lambda_j}\Phi), \Lambda_j]) = -k(A, \tilde{\rho}(\Phi, \rho_{\Lambda_j}^2\Phi)) + k(\Lambda_j, \tilde{\rho}(\Phi, \rho_A\rho_{\Lambda_j}\Phi)) \tag{60}$$

follows from (36), (38) and (39). But

$$\begin{aligned}k(\Lambda_j, \tilde{\rho}(\Phi, \rho_A\rho_{\Lambda_j}\Phi)) &\overset{(35)}{=} h(\rho_{\Lambda_j}\Phi, \rho_A\rho_{\Lambda_j}\Phi) + h(\rho_A\rho_{\Lambda_j}\Phi, \rho_{\Lambda_j}\Phi) \\ &= -h(\rho_A\rho_{\Lambda_j}\Phi, \rho_{\Lambda_j}\Phi) + h(\rho_A\rho_{\Lambda_j}\Phi, \rho_{\Lambda_j}\Phi) = 0\end{aligned}$$

since ρ_A is anti-Hermitian. Since $A \in \mathfrak{g}_o$ was chosen arbitrarily, (60) then implies that

$$[\tilde{\rho}(\Phi, \rho_{\Lambda_1}\Phi), \Lambda_1] + [\tilde{\rho}(\Phi, \rho_{\Lambda_2}\Phi), \Lambda_2] + \tilde{\rho}(\Phi, (\rho_{\Lambda_1}^2 + \rho_{\Lambda_2}^2)\Phi) = 0. \tag{61}$$

So (58), (59) and (61) imply that $\gamma' = 0$ and hence $\gamma = \text{const}$ on the interval $[r_1, r_2)$. Clearly $\gamma(r_1) = 0$ then implies that $\gamma(r) = 0$ for all $r \in [r_1, r_2)$. □

It remains to investigate the consistency of the Yang–Mills equations (45) and (46) together with (43). First we have the following proposition.

Proposition 3. *Let* $\tilde{R}_j := \tilde{\rho}(\Phi, \rho_{\Lambda_j}\Phi)$ *for* $j = 1, 2$. *Then*

$$[\tilde{R}_2, \Lambda_3] = \tilde{R}_1 \quad \textit{and} \quad [\Lambda_3, \tilde{R}_1] = \tilde{R}_2. \tag{62}$$

Proof. Suppose $A \in \mathfrak{g}_o$. Then

$$k(A, [\Lambda_3, \tilde{R}_1]) = k(A, [\Lambda_3, \tilde{\rho}(\Phi, \rho_{\Lambda_1}\Phi)]) \overset{(36),(38)}{=} -k([A, \Lambda_3], \tilde{\rho}(\rho_{\Lambda_1}\Phi, \Phi))$$
$$\overset{(39)}{=} -k(A, \tilde{\rho}(\rho_{\Lambda_3}\rho_{\Lambda_1}\Phi, \Phi)) + k(\Lambda_3, \tilde{\rho}(\rho_A\rho_{\Lambda_1}\Phi, \Phi))$$
$$\overset{(43)}{=} -k(A, \tilde{\rho}((\rho_{\Lambda_2} + \rho_{\Lambda_1}\rho_{\Lambda_3})\Phi, \Phi)) + k(\Lambda_3, \tilde{\rho}(\rho_A\rho_{\Lambda_1}\Phi, \Phi))$$
$$\overset{(44)}{=} k(A, \tilde{\rho}(\Phi, \rho_{\Lambda_2}\Phi)) + k(\Lambda_3, \tilde{\rho}(\rho_A\rho_{\Lambda_1}\Phi, \Phi)) = k(A, \tilde{R}_2) + k(\Lambda_3, \tilde{\rho}(\rho_A\rho_{\Lambda_1}\Phi, \Phi)).$$

But

$$k(\Lambda_3, \tilde{\rho}(\rho_A\rho_{\Lambda_1}\Phi, \Phi)) \overset{(35)}{=} h(\rho_{\Lambda_3}\rho_A\rho_{\Lambda_1}\Phi, \Phi) + h(\Phi, \rho_{\Lambda_3}\rho_A\rho_{\Lambda_1}\Phi)$$
$$= -h(\rho_A\rho_{\Lambda_1}\Phi, \rho_{\Lambda_3}\Phi) - h(\rho_{\Lambda_3}\Phi, \rho_A\rho_{\Lambda_1}\Phi)\,(\text{since } \rho_{\Lambda_3} \text{is anti-Hermitian}) \overset{(44)}{=} 0.$$

Since $A \in \mathfrak{g}_o$ was chosen arbitrarily, the above two results imply that $[\Lambda_3, \tilde{R}_1] = \tilde{R}_2$. Similar calculations show that $[\tilde{R}_2, \Lambda_3] = \tilde{R}_1$. □

With (62) it follows easily that (45) and (43) together imply (46). In fact, these Yang–Mills equations are more conveniently described in complex form. Let $\mathfrak{g} = \mathfrak{g}_o \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}_o$ so that $\mathfrak{g}_o$ is its compact real form with respect to the conjugation $c : \mathfrak{g} \to \mathfrak{g} : X + \mathrm{i}Y \mapsto X - \mathrm{i}Y \,\forall\, X, Y \in \mathfrak{g}_o$ and let

$$\Lambda_0 = 2\mathrm{i}\Lambda_3, \qquad \Lambda_\pm := \mp\Lambda_1 - \mathrm{i}\Lambda_2 \tag{63}$$

so that $\Lambda_- = -c(\Lambda_+)$ and $c(\Lambda_0) = -\Lambda_0$ and, by (43),

$$[\Lambda_0, \Lambda_\pm] = \pm 2\Lambda_\pm. \tag{64}$$

The Yang–Mills equations (45) and (46) are then equivalent to

$$r^2 S^{-1}(NS\Lambda_+')' - \mathrm{i}[\hat{F}, \Lambda_+] = -r^2(\tilde{R}_1 + \mathrm{i}\tilde{R}_2) =: -r^2\tilde{R}_+. \tag{65}$$

With respect to the invariant metric k the operator ad_{Λ_0} is Hermitian and $\mathfrak{g}$ can be decomposed into eigenspaces of ad_{Λ_0},

$$\mathfrak{g} = \bigoplus \mathfrak{g}_n, \quad \mathfrak{g}_n := \{X \in \mathfrak{g} : [\Lambda_0, X] = nX\}. \tag{66}$$

By (64), $\Lambda_+(r) \in \mathfrak{g}_2 \,\forall\, r$ and therefore so are Λ_+', Λ_+'', and also, by (59), $[\hat{F}, \Lambda_+]$. On the other hand, Proposition 3 implies that also the right-hand side of (65) lies in $\mathfrak{g}_2$ for any (anti-Hermitian) representation $\rho : \mathfrak{g}_o \to \mathfrak{gl}(V)$ provided that the Higgs field satisfies (44). Eq. (65) thus represent consistent second-order differential equations for the gauge potential components, subject only to the constraints (47) being satisfied at one point.

5.4. Explicit form of the field equations

If we are just trying to construct a local solution of the EYMH equations in some radial interval in which none of r, $N(r)$ and $S(r)$ is zero we can choose a constant $\Lambda_0 \in \mathfrak{g}$, subject to it being an integral lattice point within the closed fundamental Weyl chamber of some Cartan subalgebra and satisfying $c(\Lambda_0) = -\Lambda_0$. (This will fix an explicit action of the symmetry group $K_o = SU(2)$ by automorphisms on the principal bundle [15].) Then any r-dependent $\Lambda_+ \in \mathfrak{g}$ may be chosen subject to (64) and, at one point, to (47).

But the interesting and physically more relevant EYMH fields are global ones which remain regular at the center $r = 0$ or at a black hole horizon where $N = 0$ and which have an appropriate asymptotic behavior. It is clear from the expressions for energy density and pressures in Eqs. (53)–(55) that $\varLambda_1'$, $\rho_{\varLambda_1}\varPhi$ and, in particular, $\hat{F}$ must vanish for $r = 0$. This means that also $[\varLambda_1, \varLambda_2] = \varLambda_3$ at that point which in turn implies that the induced Lie algebra homomorphism $\lambda' : \mathfrak{k}_0 \to \mathfrak{g}_o$ defines a so-called A_1 (or defining) vector $\Lambda_0 = 2\mathrm{i}\,\varLambda_3$ in the Cartan subalgebra of the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_o \otimes \mathbb{C}$ and thus a conjugacy class of $\mathfrak{sl}(2)$-subalgebras. Even when no regularity at the center is required, for example when solutions need only be found outside a black hole, natural physical fall-off conditions at infinity also imply $\hat{F} = 0$ (at least when the space-time is asymptotically flat and the magnetic charge vanishes). We will therefore from now on make the assumption that Λ_0 is an A_1-vector.

Up to conjugacy these A_1-vectors and their corresponding subalgebras form a finite set and, given a base $\{\alpha_1, \ldots, \alpha_\ell\}$ of the set of roots R of the Lie algebra $\mathfrak{g}$, are uniquely described by the characteristic $\chi = (\alpha_1(\Lambda_0), \ldots, \alpha_\ell(\Lambda_0))$. There is always a root base $\varDelta$ such that $\alpha_k(\Lambda_0) \in \{0, 1, 2\}\,\forall\, k$, and all possible characteristics and thus all conjugacy classes of $\mathfrak{sl}(2)$-subalgebras of simple Lie algebras have been classified [17,28]. In view of (64) the invariant connection on the principal bundle for a given conjugacy class of K-actions of automorphisms is then fully given by the (complex) functions $w_\alpha(r)$ such that

$$\Lambda_+ = \sum_{\alpha \in S_\lambda} w_\alpha \mathbf{e}_\alpha, \qquad S_\lambda := \{\alpha \in R : \alpha(\Lambda_0) = 2\}. \tag{67}$$

Here we have introduced a Chevalley–Weyl basis $\{\mathbf{h}_\alpha, \mathbf{e}_\beta, \mathbf{e}_{-\beta} : \alpha \in \varDelta, \beta \in R^+\}$ (where R^+ is the set of positive roots, cf., for example [29]) of $\mathfrak{g}$ for which we adopt the conventions and definitions[1]

$$[\mathbf{e}_\alpha, \mathbf{e}_{-\alpha}] = \mathbf{h}_\alpha, \qquad [\mathbf{e}_\alpha, \mathbf{e}_\beta] = \nu_{\alpha,\beta}\,\mathbf{e}_{\alpha+\beta}, \qquad \nu_{-\alpha,-\beta} = -\nu_{\alpha,\beta} \text{ or } 0, \quad \text{if } \alpha + \beta \notin R, \tag{68}$$

$$|\alpha|^2 := k(\alpha, \alpha), \qquad \langle \alpha, \beta \rangle := \frac{2k(\alpha, \beta)}{|\beta|^2} \quad \forall\, \alpha, \beta \in R, \tag{69}$$

$$k(\mathbf{e}_\alpha, \mathbf{e}_{-\alpha}) = -2|\alpha|^{-2} \quad \forall\, \alpha, \beta \in R, \qquad (c_{ij}) := (\langle \alpha_i, \alpha_j \rangle) \quad \text{(Cartan matrix)}. \tag{70}$$

The gauge connection is thus described by as many complex functions of r as there are elements in S_λ. In fact, by the definition of the roots, the eigenspace $\mathfrak{g}_2$ of ad_{Λ_0} is spanned by the set $\{\mathbf{e}_\alpha : \alpha \in S_\lambda\}$.

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[1] If the gauge group is semisimple and an invariant inner product on $\mathfrak{g}$ contains more than one 'coupling' constant this may have to be modified.

In the EYM case the field equations need to be solved for the two real functions N (or m) and S of r and the complex functions $w_\alpha = \omega_\alpha \mathrm{e}^{\mathrm{i}\gamma_\alpha} = u_\alpha + \mathrm{i}v_\alpha$ for $\alpha \in S_\lambda$. This turns out to be considerably simpler if the set S_λ forms a Π-system [17], i.e. if $\alpha, \beta \in S_\lambda$ implies that $\alpha - \beta$ is not a root. Then $[\mathbf{e}_\alpha, \mathbf{e}_{-\beta}] = 0$ if α and β are two distinct elements of S_λ [15,16]. Then S_λ also generates a subalgebra of $\mathfrak{g}$. In particular, S_λ is a Π-system if Λ_0 is contained in the open Weyl chamber of the Cartan subalgebra of $\mathfrak{g}$ [15] which means, in particular, that $\alpha(\Lambda_0) > 0\,\forall\,\alpha \in R^+$. We have called this the *regular case*.

The simplification occurs largely because the constraint equation (47) then implies that the phase γ_α of w_α is constant and can be chosen zero by a gauge choice. As the following shows this may not always be the case in the presence of Higgs fields, but the equations are still much simpler.

In the following we will derive an explicit form for the Yang–Mills and the Higgs equations only since no new insight is gained by reformulating Einstein's equations.

The left-hand side of Eqs. (45)–(47) has been derived in [16,18]. From (67) we have

$$r^2 S^{-1}(NSw'_\alpha)' + \frac{1}{2}\left(\alpha(\Lambda_0)w_\alpha - \sum_{\beta\in S_\lambda}\langle\beta,\alpha\rangle|w_\alpha|^2 w_\beta + \sum_{\beta,\gamma,\delta\in S_\lambda}\mu_{\alpha\delta\beta\gamma}w_\beta\bar{w}_\gamma w_\delta\right)$$
$$= -r^2\tilde{R}_{+,\alpha} \quad \forall\,\alpha \in S_\lambda \tag{71}$$

and

$$\sum_{\alpha,\beta\in S_\lambda}(w_\alpha\bar{w}'_\beta - w'_\alpha\bar{w}_\beta)[\mathbf{e}_\alpha, \mathbf{e}_{-\beta}] = 2r^2\tilde{\rho}(\Phi, \Phi'), \tag{72}$$

where

$$[\mathbf{e}_\alpha, [\mathbf{e}_\beta, \mathbf{e}_{-\gamma}]] =: \sum_{\delta\in S_\lambda}\mu_{\delta\alpha\beta\gamma}\mathbf{e}_\delta$$

and $\tilde{R}_{+,\alpha}$ is the $\mathbf{e}_\alpha$-component of $\tilde{R}_+$. Note that it follows from Proposition 3 that $\tilde{R}_+ \in \mathfrak{g}_2 = \mathrm{span}\{\mathbf{e}_\alpha : \alpha \in S_\lambda\}$. Moreover, $[\mathbf{e}_\alpha, \mathbf{e}_{-\beta}] \in \mathfrak{g}_0$ if $\alpha, \beta \in S_\lambda$ and $\tilde{\rho}(\Phi, \Phi') \in \mathfrak{g}_0$. By Proposition 2, Eq. (72) needs to be solved for the w'_α's only for one r-value.

In the *regular case* $\mu_{\alpha\beta\gamma\delta} = 0$, so (71) represents the components of an equation in the span of $\{\mathbf{h}_\alpha : \alpha \in S_\lambda\}$. Moreover, $[\mathbf{e}_\alpha, \mathbf{e}_{-\beta}] = \delta_{\alpha\beta}\mathbf{h}_\alpha$ so that (72) becomes a condition for the derivatives of the phases of the complex functions $w_\alpha(r)$ (which when the right-hand side vanishes like in the EYM case means that the phases will be constant and the w_α's can be chosen real by fixing the gauge.)

Since in (35) the quantity $\tilde{\rho}$ is only defined for $A \in \mathfrak{g}_o$ in order to evaluate the right-hand side of (71) and (72) we introduce (temporarily) the basis

$$\hat{\mathbf{h}}_j := -\tfrac{\mathrm{i}}{2}\mathbf{h}_j, \qquad \hat{\mathbf{e}}_\alpha := \tfrac{1}{2}(-\mathbf{e}_\alpha + \mathbf{e}_{-\alpha}), \qquad \hat{\mathbf{f}}_\alpha := \tfrac{\mathrm{i}}{2}(\mathbf{e}_\alpha + \mathbf{e}_{-\alpha}) \quad (j = 1, \ldots \ell, \alpha \in R^+), \tag{73}$$

whose $\mathbb{R}$-linear span is the compact real form $\mathfrak{g}_o$ of $\mathfrak{g}$. In this basis $\{\mathbf{e}_\Gamma\} = \{\mathbf{h}_j, \hat{\mathbf{e}}_\alpha, \hat{\mathbf{f}}_\alpha\}$ the invariant metric then has the form

$$(\hat{k}_{\Gamma\Delta}) = \begin{pmatrix} \frac{1}{2}|\alpha_i|^{-2}c_{ij} & 0 & 0 \\ 0 & |\alpha|^{-2}\delta_{\alpha\beta} & 0 \\ 0 & 0 & |\alpha|^{-2}\delta_{\alpha\beta} \end{pmatrix}. \tag{74}$$

We extend the anti-Hermitian representation $\rho : \mathfrak{g}_o \to \mathfrak{gl}(V)$ to $\mathfrak{g}$ in the obvious way, $\rho_{X+\mathrm{i}Y} := \rho_X + \mathrm{i}\rho_Y$, and let

$$\rho_j := \rho_{\mathbf{h}_j} \,\forall\, j = 1, \ldots, \ell \quad \text{and} \quad \rho_\alpha := \rho_{\mathbf{e}_\alpha} \,\forall\, \alpha \in R^+. \tag{75}$$

It then follows for the Hermitian conjugates with respect to the inner product h on V that

$$\rho_j^+ = \rho_j, \qquad \rho_\alpha^+ = \rho_{-\alpha} \tag{76}$$

and that ρ_{Λ_0} is Hermitian and $\rho_{\Lambda_\pm}^+ = \rho_{\Lambda_\mp}$. Denoting the inverse of $k(\hat{\mathbf{h}}_i, \hat{\mathbf{h}}_j)$ by

$$\hat{k}^{ij} = 2c^{ij}|\alpha_j|^2, \tag{77}$$

where (c^{ij}) is the inverse of the Cartan matrix, we find from (35) that

$$\begin{aligned}\tilde{\rho}(x, y) = &- \sum_{i,j=1}^{\ell} \hat{\mathbf{h}}_i \, \hat{k}^{ij} \, \mathbf{Im} \, h(\rho_j x, y) - \sum_{\alpha \in R^+} |\alpha|^2 (\mathbf{Re}[h(\rho_\alpha x, y) - h(x, \rho_\alpha y)]\hat{\mathbf{e}}_\alpha \\ &- \mathbf{Im}[h(\rho_\alpha x, y) + h(x, \rho_\alpha y)]\hat{\mathbf{f}}_\alpha).\end{aligned} \tag{78}$$

Now any (finite-dimensional) representation of G is the direct sum of irreducible ones which can be obtained from irreps of $\mathfrak{g}$ and are characterized by their highest weight $\Lambda \in \mathfrak{h}^*$ (the dual of the Cartan subalgebra $\mathfrak{h}$). Any other weight μ is then given by $\mu = \Lambda - \sum_{i=1}^{\ell} q_i \alpha_i$ for certain nonnegative integers q_i. The set of eigenvalues of ρ_{Λ_0} is $\mathcal{E}_o = \{\mu_k(\Lambda_0)\}$, where the μ_k are the weights of the representation. Thus spherically symmetric Higgs fields for a given Λ_0, i.e. choice of the action of K, and a given representation ρ exist provided at least one of the irreducible components of ρ has a weight μ with $\mu(\Lambda_0) = 0$.

In particular, for the adjoint representation, $\rho = \mathrm{ad}$ and $V = \mathfrak{g}_o$, there is always a weight 0 with multiplicity equal to the rank of $\mathfrak{g}$ and any $\Phi \in \mathfrak{h}$ is a solution of (44).

Moreover, in the *regular case* where $\alpha(\Lambda_0) > 0$ for all positive roots α every solution of (44) lies in $\mathfrak{h}$.

Next we observe that since ρ_{Λ_0} is a Hermitian operator the vector space V is a direct sum of mutually orthogonal eigenspaces of ρ_{Λ_0},

$$V = \bigoplus_{\sigma \in \mathcal{E}_o} \hat{V}_\sigma, \quad \hat{V}_\sigma := \{x \in V : \rho_{\Lambda_0} x = \sigma x\} \tag{79}$$

so that (44) now states that $\Phi(r) \in \hat{V}_0 \,\forall\, r$ and thus also $\Phi'(r) \in \hat{V}_0$. Moreover, it follows easily that

$$\rho_\alpha V_\sigma \subset V_{\sigma + \alpha(\Lambda_0)} \quad \text{and} \quad \rho_h V_\sigma \subset V_\sigma \quad \forall\, h \in \mathfrak{h} \tag{80}$$

and therefore $\rho_\alpha \Phi \in \hat{V}_2$ when $\alpha \in S_\lambda$ so that, in particular,

$$\rho_{\Lambda_\pm} \Phi \in \hat{V}_{\pm 2} \quad \text{and} \quad \rho_j \Phi \in \hat{V}_0. \tag{81}$$

Thus, if we replace x and y in (78) by Φ and Φ', respectively, we get the expression needed on the right-hand side of (72) except that the second sum needs only be taken over those roots $\alpha \in R^+$ for which $\alpha(\Lambda_0) = 0$, in view of (80), since both Φ and Φ' lie in $\hat{V}_0$ and the $\hat{V}_\sigma$ are orthogonal for distinct σ.

In the *regular case* $\alpha(\Lambda_0) > 0\,\forall\,\alpha \in S_\lambda$ so that the constraint equation (72) becomes

$$\sum_{j=1}^{\ell} \hat{k}_{ij} a_j |\alpha_j|^2 |\alpha|^{-2} \omega_\alpha^2 \gamma'_\alpha = -\tfrac{1}{2} r^2 \,\mathbf{Im}\, h(\rho_i \Phi, \Phi') \quad \forall\, i = 1, \ldots, \ell,\ \forall\, \alpha \in S_\lambda, \tag{82}$$

where $\alpha = \sum_{j=1}^{\ell} a_j \alpha_j$.

In the evaluation of $\tilde{R}_+$ one obtains expressions $h(\rho_{\pm\alpha}\Phi, \rho_{\pm\beta}\Phi)$ (for all choices of the signs) where $\alpha \in S_\lambda$ and $\beta \in R^+$. But if $\Phi \in \hat{V}_0$ then $\rho_{\pm\alpha}\Phi \in \hat{V}_{\pm 2}$ and $\rho_{\pm\beta}\Phi \in \hat{V}_{\pm\beta(\Lambda_0)}$. Since these eigenspaces of ρ_{Λ_0} are mutually orthogonal the only inner products that are nonzero are those for which $\alpha, \beta \in S_\lambda$ and the signs are the same. It follows that

$$\tilde{R}_+ = \sum_{\alpha \in S_\lambda} \tilde{R}_{+,\alpha} \mathbf{e}_\alpha = \sum_{\alpha \in S_\lambda} |\alpha|^2 Q_\alpha \mathbf{e}_\alpha, \tag{83}$$

where

$$Q_\alpha := \frac{1}{2} \sum_{\beta \in S_\lambda} (R_{\alpha,\beta} + \bar{R}_{-\alpha,-\beta}) w_\beta \tag{84}$$

and

$$R_{\alpha,\beta} := h(\rho_\alpha \Phi, \rho_\beta \Phi), \quad \alpha, \beta \in R,\ \Phi \in \hat{V}_0. \tag{85}$$

Again, in the *regular case*, or whenever we know that $\alpha, \beta \in S_\lambda$ implies that $\alpha - \beta$ is not a root, this simplifies somewhat. For we have $\rho_{-\alpha}\rho_\beta\Phi = \rho_{[\mathbf{e}_{-\alpha},\mathbf{e}_\beta]}\Phi + \rho_\beta\rho_{-\alpha}\Phi = \rho_\beta\rho_{-\alpha}\Phi$ and therefore $R_{-\alpha,-\beta} = h(\rho_{-\alpha}\Phi, \rho_{-\beta}\Phi) = h(\Phi, \rho_\alpha\rho_{-\beta}\Phi) = h(\Phi, \rho_{-\beta}\rho_\alpha\Phi) = h(\rho_\beta\Phi, \rho_\alpha\Phi) = \overline{h(\rho_\alpha\Phi, \rho_\beta\Phi)} = \bar{R}_{\alpha,\beta}$ so that

$$Q_\alpha = \sum_{\beta \in S_\lambda} R_{\alpha,\beta} w_\beta. \tag{86}$$

6. Conclusions

We have shown how to, in principle, construct Einstein–Yang–Mills–Higgs systems that are invariant under an arbitrary action of a space-time symmetry group that acts by principle bundle automorphisms which leave the gauge connection invariant as well as Higgs fields defined via any unitary representation of the (compact) gauge group. The classification of the possible actions by automorphisms is known for the symmetry group $SU(2)$, but may be more difficult to find for larger groups. One would need to first find all conjugacy classes of a certain type of Lie subalgebras of the gauge Lie algebra.

We have obtained an explicit form of the full field equations in the static spherically symmetric case and shown that they form a consistent system of ordinary differential equations. Before global solutions can be found numerically it would be necessary to investigate in some detail the boundary conditions that regularity conditions at a center, horizon or in an asymptotic region will imply.

It must be pointed out that not all cases of physical interest even for the static spherically symmetric case are covered by this approach. For example, the doublet Higgs field coupled to an $SU(2)$-gauge and gravitational field in [30] cannot be described in our formalism because the Higgs field is not spherically symmetric. In fact, the representation of $\mathfrak{su}(2)$ in this case is the direct sum of two irreducible two-dimensional ones for which there is no weight μ with

$\mu(\Lambda_0) = 0$. These authors make a simple ansatz for the Higgs field using the gauge choice for the potential often attributed to Witten [26]. They then find that the stress-energy tensor is spherically symmetric and thus compatible with a spherically symmetric ansatz for the space-time metric.

One might ask whether with our gauge choice one can assume that only the quantity $h(D_{(\alpha}\Phi, D_{\beta)}\Phi)$ is spherically symmetric rather than Φ itself. Unfortunately this is not possible since then $h(\rho_{\Lambda_3}\Phi, \rho_{\Lambda_3}\Phi)$ would have to vanish which implies $\rho_{\Lambda_3}\Phi = 0$, i.e. an invariant Higgs field. On the other hand we do not know whether the Witten ansatz for spherically symmetric gauge fields can be generalized to gauge groups other than $SU(2)$ or whether perhaps another equally convenient gauge exists.

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